Norm-Attaining Operators

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Notations

Before we start, let us see the notations first.

$\mathbb{K}$	the field of real or complex scalars
ℓ_1	the set of absolutely summable sequences
ℓ_2	the set of square summable sequences
ℓ_∞	the set of bounded sequences
c_0	the set of convergent sequences coverging to 0
B_X	the closed unit ball in X
S_X	the unit sphere in X
$\overline{M}$	the closure of M
X^*	the dual of X
T^*	the adjoint of T
$\|T\|$	a norm of the operator T
$B(X, Y)$	the space of bounded linear operators from X into Y

Outline of the talk

The theory of norm-attaining operators is a very recent field of research, as it appeared in the second half of the 20th Century, and it is also very active nowadays, since many authors contribute to this field with their research now.

In spite of its short life, the work in the theory of norm-attaining operators has been very fruitful, and there exist many important results related to this field.

We shall discuss the following in the lecture.

- Norm-attaining functionals in X^*.
- Non-norm-attaining functionals in X^*.
- Norm-attaining operators.
- Operators that attain their minima.

Let X and Y be normed spaces and let $T \in B(X, Y)$.

The **operator norm** is given by

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|.$$

Definition 1.

Let $T \in B(X, Y)$. The operator T is said to be **norm-attaining** if there exists $x_0 \in S_X$ such that

$$\|Tx_0\| = \|T\|.$$

We denote the set of all norm-attaining operators from X to Y by $NA(X, Y)$ and $NA(X, X) = NA(X)$. We shall see that $NA(X, Y)$ is not a subspace of $B(X, Y)$ in general.

Problem of density : When is $\overline{NA(X, Y)} = B(X, Y)$?

Introduction

The problem of density of norm-attaining operators involves two spaces, the spaces of the domain and codomain of the operators under study.

The problem has its origin in an important result of the theory of Banach spaces, the **Bishop-Phelps Subreflexivity Theorem**.

Although it appeared just half a century ago (more specifically, in 1961), it is considered **a classical result of Functional Analysis**. This theorem states that **every Banach space is subreflexive** (that is, the set of norm-attaining operators in $X^* = B(X, \mathbb{K})$ for a Banach space X is dense in the dual space X^*). Proof will not be discussed in the lecture.

This result appeared in 1961, although it was improved in 1963 by the same authors, with a generalization involving support functionals. In 1970, B. Bollobas made some quantitative changes to this result, obtaining what we know nowadays as the **Bishop-Phelps-Bollobas Theorem**.

Introduction

In the light of the Bishop-Phelps subreflexivity theorem, Bishop and Phelps posed the following question in the research paper (appeared in 1961) :

First Question : **What conditions on two Banach spaces X and Y assure that the collection $NA(X, Y)$ of norm-attaining operators is dense in $B(X, Y)$?**

There must be some additional conditions imposed on at least one of the spaces; in particular, in a 1963, paper **Joram Lindenstrauss** showed a counterexample that there is a Banach space X such that the set $NA(X)$ of norm-attaining operators is not dense in $B(X)$. Many other counterexamples have been developed later.

The seminal paper of Lindenstrauss constitutes one of the biggest contributions to this field of research. He did not only show that the expression $\overline{NA(X,Y)} \neq B(X,Y)$ in general, but also gave a slightly weaker affirmation which holds for every Banach spaces X and Y.

Joram Lindenstrauss, 1975-2012

Introduction

Since Bishop and Phelps showed that the norm-attaining operators are dense in $B(X, Y)$ **for every Banach space** X **and** $Y = \mathbb{K}$, it is particularly natural to ask the following question :

Second Question : **What conditions on a Banach space Y would assure that the norm-attaining operators are dense in $B(X, Y)$ for every Banach space X.**

V. Zizler in 1973 proved that the expression $\overline{NA(X, Y)} = B(X, Y)$ is true when X is a reflexive space and Y is an arbitrary Banach space.

Thus **reflexive spaces are positive examples for the problem of the density.**

Introduction

One of the reasons of the importance of the problem of density of norm-attaining operators is its intimate conection with another important subject in Banach spaces, the Radon-Nikodym property (RNP).

In fact, the study of norm-attaining operators provides certain familiarity with the geometric aspects of the RNP, like the dentability.

Norm-attaining Functionals

Let X be a normed space and $f \in X^*$. Then, we define the norm of f by

$$\|f\| = \sup\{|f(x)| : x \in B_X\}.$$

Definition 1.

We say that a bounded linear functional f **attains its norm** *when the supremum in the previous definition of norm is a maximum, i.e., if there exists $x_0 \in B_X$ such that*

$$\|f\| = |f(x_0)|.$$

Norm-attaining Functionals

Let us see some examples of norm-attaining functionals in ℓ_1^* and c_0^*, respectively.

Example 2.

Consider the functional $f : \ell_1 \to \mathbb{K}$ *defined by*

$$x = \{x_n\} \mapsto \sum_{n=1}^{\infty} \frac{x_n}{n}.$$

Then f *is a bounded linear functional and attains its norm.*

Example 3.

Consider the functional $g : c_0 \to \mathbb{K}$ *defined by*

$$x = \{x_n\} \mapsto x_1 + x_2.$$

Then g *is a bounded linear functional and attains its norm.*

Norm-attaining Functionals

We want to study the existence of this kind of functionals for every Banach space X. As a consequence of Hahn-Banach theorem, we have the following corollary :

Corollary 4.

For every $x \in X$, there exists $f \in X^*$ verifying $\|f\| = 1$ and $f(x) = \|x\|$.

Using this corollary in the particular case of the unit sphere, we get that for every $x_0 \in S_X$ there exists $f \in X^*$ with $\|f\| = 1$ and $f(x_0) = \|x_0\| = 1$.

Consequently, **for every Banach space, there exist functionals which attain their norm**.

Norm-attaining Functionals

Let X be a Banach space.

- How many functionals in X attain their norm?

Bishop-Phelps subreflexivity theorem shows that for every Banach space X the set of norm-attaining functionals is quite big, it is dense in the set of bounded linear functionals. That is, $\overline{NA(X, \mathbb{K})} = X^*$.

- Does there exist a Banach space where every functional attains its norm?

Before answering the question, we now see examples that there are Banach spaces which contain some non-norm-attaining functionals.

Example 5.

Consider the functional $f : \ell_1 \to \mathbb{K}$ defined by

$$x = \{x_n\} \mapsto \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x_n.$$

Then f is a bounded linear functional but does not attain its norm.

Non-Norm-attaining Functionals

Example 6.

Consider the functional $g : c_0 \to \mathbb{K}$ *defined by*

$$x = \{x_n\} \mapsto \sum_{n=1}^{\infty} 2^{-n+1} x_n.$$

Then g *is a bounded linear functional but does not attain its norm.*

Norm-attaining Functionals

There exists a certain class of spaces, the **reflexive spaces**, such that every bounded linear functional in these spaces attains its norm. That is, if X is reflexive, then $NA(X, \mathbb{K}) = X^*$.

Moreover, this fact is a characterization of reflexivity. This result is the **James Theorem** and constitutes another classical result of Functional Analysis.

Theorem 7 (James Theorem).

A Banach space X is reflexive if and only if every continuous linear functional on X attains its maximum on the closed unit ball in X.

Proof will not be discussed in the lecture.

We have seen some examples of non-norm-attaining functionals. We used before, ℓ_1 and c_0 (we chose these spaces as they are some of the most-known classical Banach spaces and are not reflexive).

Reflexive Spaces

We say that a Banach space is reflexive if it coincides with its bidual space. More specifically, if we denote the dual space of X by X^*, and the bidual space by X^{**}, and consider for every $x \in X$ the function $J(x) : X^* \to \mathbb{K}$ given by

$$J(x)(f) = f(x) \qquad f \in X^*,$$

then $J(x) \in X^{**}$, so we obtain a map $J : X \to X^{**}$ called the **evaluation map**. From the Hahn- Banach theorem, J is injective and preserves norms.

A Banach space is **reflexive** when its evaluation map is surjective.

We have seen that, for every Banach space, there exist norm-attaining functionals, and that for some spaces, reflexive spaces, every functional attains its norm.

Norm-attaining operators defined on finite dimensional spaces

Let X and Y be normed spaces, $\dim(X) < \infty$ and let $T \in B(X, Y)$. The operator norm is given by

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|.$$

It is well-known that if X has finite dimension, then the closed unit ball in X is compact (Heine-Borel Theorem) and the above "supremum" is a maximum. That is, if $\dim(X) < \infty$, there exists $x_0 \in B_X$ such that

$$\|T\| = \|Tx_0\|.$$

Theorem 8.

Let X and Y be normed spaces and $\dim(X) < \infty$. *Then* $NA(X, Y) = B(X, Y)$.

Norm-attaining operators

We saw that for every Banach space there exist norm-attaining functionals in its dual space, as a consequence of the Hahn-Banach theorem. This result can be considered the first natural example of norm-attaining operators. However, as the Hahn-Banach theorem is only valid with the field as the codomain, it is clear that we cannot use it now to guarantee the existence of norm-attaining operators.

In fact, it remains an open question if for every X and Y Banach spaces there exists a norm-attaining operator from X to Y (a non semi-trivial one). However, we can study the density of the set of norm-attaining operators in $B(X, Y)$.

Before studying the density of these operators, we can show some examples of specific operators which attain their norm and others that do not.

For the first example, we want to show an operator that does not attain its maximum in the unit ball.

Example 9.

We consider $c = \{c_n\} \in \ell_\infty$ *such that* $|c_n| < \sup |c_n|$ *(for example, take* c_n *positive and growing to* 1*) and the operator* $T : \ell_2 \to \ell_2$ *given by*

$$T(x) = cx = \{c_n x_n\}, \qquad \text{for all} \quad x = \{x_n\} \in \ell_2.$$

The operator T *is bounded but it does not attain its norm.*

In the following example, we will study the Fourier coefficients associated to a function.

Example 10.

Define $T : L_1(\mathbb{T}) \to c_0$ *by*

$$T(f) = \{\tilde{f}(n)\}$$

where $\{\tilde{f}(n)\}$ *is the sequence of Fourier coefficients associated to* $f \in L_1(\mathbb{T})$. *Here* $\|T\| = 1$ *and* T *attains its norm.*

Norm-attaining operators on Hilbert spaces

Let H and K be Hilbert spaces. We denote the set of all norm-attaining operators from H to K by $NA(H, K)$ and $NA(H, H) = NA(H)$.

We have seen that if $\dim(H) < \infty$, then any $T \in B(H, K)$ is norm-attaining.

Although, in infinite dimension space this important property is lost, albeit it remains for the compact operators.

Theorem 11.

Every compact operator is norm-attaining.

We recall the following.

- Let X be a Banach space. If X is reflexive, then the closed unit ball in X is sequentially compact in the weak topology.
- Let X, Y be Banach spaces and $T : X \to Y$ be compact. If $x_n \to x$ weakly, then $Tx_n \to Tx$ strongly.

Exercises

Definition 12.

An operator $S \in B(H, K)$ *is called an* **isometry** *if* $\|Sx\| = \|x\|$ *for all* $x \in H$.

Exercise 13.

- *Identity operator is norm-attaining.*
- *Every isometry is norm-attaining.*

Example 14.

Let $\{e_n\}$ be a an orthonormal basis in ℓ_2. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that

1. $0 < a_1 < a_2 < \dots$;
2. $a_n \to a$, *for some* $a \in (0,1]$;
3. $a_n^2 + b_n^2 = 1$.

Let T be the unitary operator given by

$$Te_n = (a_n + ib_n)e_n, \qquad n = 1, 2, 3, \dots .$$

Then $(T + I)$ is not norm-attaining.

Here $\|T + I\| = 2\sqrt{1+a}$ *and* $\|(T+I)x\| < \|T + I\|$ *for any* $x \in S_X$.

Properties of norm-attaining operators

Theorem 15 ([1]).

1. *Let* $T \in B(H)$ *be self-adjoint. Then* $T \in NA(H)$ *iff* $\|T\|$ *or* $-\|T\|$ *is an eigen value of* T.
2. *Let* $T \in B(H)$ *be a positive operator. Then* $T \in NA(H)$ *iff* $\|T\|$ *is an eigen value of* T.
3. *Let* H *and* K *be complex Hilbert spaces and* $T \in B(H, K)$. *Then* $T \in NA(H)$ *iff* $T^*T \in NA(H)$.

Properties of norm-attaining operators

Definition 16.

An operator $P \in B(H)$ is called **positive** *if $\langle Px, x\rangle \geq 0$, for all $x \in H$.*

Given an operator $T \in B(H, K)$, we denote by P_T, the unique operator called the **positive square root** of T^*T, that is, $\langle P_T x, x\rangle \geq 0$, for all $x \in H$ and $P_T^2 = T^*T$.

Theorem 17.

1. *Let $T \in B(H, K)$. Then $T \in NA(H)$ iff $P_T \in NA(H)$.*
2. *Let $T \in B(H, K)$. Then $T \in NA(H, K)$ iff $T^* \in NA(K, H)$.*
3. *Let $T \in NA(H, K)$. If there exists $x_0 \in S$ such that $\|Tx_0\| = \|T\|$, then $T(s_0^{\perp}) \subseteq (Tx_0)^{\perp}$ (span$\{x_0\}$ reduces T).*

Properties of norm-attaining operators

Theorem 18 ([3]).

Let H and K be complex Hilbert spaces and $T \in B(H,K)$. The following are equivalent :

1. $T \in NA(H,K)$.
2. $T^* \in NA(H,K)$.
3. $\||T|\|$ *is an eigenvalue of* $|T| := \sqrt{T^*T}$.
4. $\|T\|$ *is an eigenvalue of* $|T|$.
5. $|T| \in NA(H,K)$.
6. $|T^*| \in NA(H,K)$.
7. $|T|^2$ *is norm-attaining.*
8. $|T^*|^2$ *is norm-attaining.*
9. $\|T\|$ *is an eigenvalue of* $|T^*|$.

Operators that attain their minima

The study of bounded linear operators that attain their minima have some similarities with the ones that achieve their norm.

We now discuss results on operators that attain their minima. In order to have some common notations, let us call that the operator T **satisfies the property** N when it is a norm-attaining operator.

Let H and K be Hilbert spaces. The space $B(H, K)$ is a Banach space with the norm

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\|.$$

Operators that attain their minima

Analogous to norm-attaining operators, we now define the following value

$$[T] = \inf_{\|x\|=1} \|Tx\|$$

and ask when such an infimum is a minimum, which motivates the following definition.

Definition 19 ([2]).

An operator $T \in B(H, K)$ is called to satisfy the **property** N^* *if there exists an element x_0 in the unit sphere such that*

$$[T] = \|Tx_0\|.$$

Important characteristics of the N^*-operators

Exercise 20.

1. *If T is non-injective, then T attains its minimum and further* $[T] = 0$.
2. *An operator with zero minimum on the unit sphere should be non-injective in order to satisfy the property N^*. Equivalently, if T is injective and satisfies the property N^*, then* $[T] > 0$.
3. *Let $T \in B(H, K)$ with $\dim H \infty$. Then T satisfies the property N^*. Moreover,*
   (a) *if $\dim R(T) = \text{dim } H$, then $[T] > 0$;*
   (b) *if $\dim R(T) < \text{dim } H$, then $[T] = 0$.*
4. *Let $T \in B(H, K)$, with $\text{dim } H < \infty$ or $\dim K < \infty$, then T satisfies the property N^*.*

Complete characterization of N^* for non-injective compact operators

We have the complete characterization of the property N^* for non-injective compact operators. It is observed that injectiveness is an important property with respect to the property N^*.

Theorem 21.

Let $T \in B(H, K)$ be a compact operator with $\dim H = \infty$. Then T satisfies N^* iff T is non-injective.

Important characteristics of the N^*-operators

Theorem 22.

1. *If T is a self-adjoint operator on H, then for any $x \in H$ we have*
$$\|Tx\|^2 \geq [T]\langle Tx, x\rangle.$$
2. *Let $P \in B(H)$ be a positive operator. Then*
$$[P] = \inf_{\|x\|=1} \langle Px, x\rangle.$$

Important characteristics of the N^*-operators

Give $T \in B(H)$, it is well-known that

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|.$$

But

$$[T] = \inf_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|$$

is not true, which is illustrated in the following example.

Example 23.

Let $T : \ell_2 \to \ell_2$ be defined by $T(x_n) = (\lambda_n x_n)$ with $\lambda_1 > \lambda_2 > \cdots > \lambda > 0$ and $\lambda_n \to \lambda$. Then $T \geq 0$ and T does not satisfy the property N^*. But

$$\inf_{\|x\|=\|y\|=1} |\langle Tx, y\rangle| = 0.$$

Important characteristics of the N^*-operators

Exercise 24.

1. *If* $P \geq 0$ *and* $[P] = \|P\|$, *then* $P = [P]I$.
2. *If* $P \geq 0$, *then* $P^n \geq 0$ *and* $\|P^n\| = \|P\|^n$ *for all* $n \geq 1$.
3. *Let* $P \in B(H)$ *be a positive operator. Then* $[P^n] = [P]^n$.

Similarly to the property N, we have the following.

Exercise 25.

Let $P \in B(H)$ *be a positive operator. Then* P *satisfies* N^* *iff* $[P]$ *is an eigenvalue of* P.

An Example

The next example is an injective operator, which does not satisfy the N^* property.

Example 26.

Consider $T : \ell_2 \to \ell_2$ *defined by* $T(x_n) = (\lambda_n x_n)$, *where* $\lambda_1 > \lambda_2 > \cdots$ *and* $\lambda \to \lambda > 0$. *Here* T *is injective and* $T \geq 0$ *but* T *does not satisfy the* N^**-property. Note that the numerical range of* T *is the interval* $(\lambda, \lambda_1]$ *and* $[T] = \lambda$ *is not an extreme point of the numerical range.*

Examples

Given an operator T on H which satifies the N-condition, it is not necessary true that T^2 also satisfies N. In a similar way, it may happen that an operator satisfies N^* but T^2 does not satisfy N^*.

Example 27.

Let $T : \ell_2 \to \ell_2$ *by* $T(x_1, x_2, \ldots) = (\lambda x_2, 0, \lambda_1 x_3, \lambda_2 x_4, \ldots)$.

1. *If we choose* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda$ *and* $\lambda_n \to \lambda$, *then* T *satisfies the property* N *but* T^2 *does not satisfy the property* N.
2. *If we choose* $\lambda_1 > \lambda_2 > \cdots > \lambda > 0$ *and* $\lambda_n \to \lambda$, *then* T *satisfies the property* N^* *but* T^2 *does not satisfy the property* N^*.

Results on properties N and N^*

Theorem 28.

Let $P \in B(H), P \geq 0$ and n be a positive integer.

1. *P satisfies N iff P^n satisfies N.*
2. *P satisfies N^* iff P^n satisfies N^*.*

Theorem 29.

Let $P \in B(H), P \geq 0$ and n, k be positive integers. Define

$$T_n = \|P\|^n I - P^n, \qquad \tilde{T}_n = P^n - [P]^n I.$$

1. *P satisfies N^* iff T^n satisfies N iff T_n^k satisfies N.*
2. *P satisfies N iff $\tilde{T}^{n^*}$ satisfies N^* iff $(T_n^*)^k$ satisfies N^*.*

Recent research

- Operators (compact / non-compact) which cannot be approximated by norm-attaining ones (counterexamples)
- $\mathcal{AN}$ operators (spectral decomposition, structure theorem)
- $\mathcal{AN}^*$ operators

References

Carvajal, X., and Neves, W.
Operators that achieve the norm.
Integral Equations Operator Theory 72, 2 (2012), 179–195.

Carvajal, X., and Neves, W.
Operators that attain their minima.
Bull. Braz. Math. Soc. (N.S.) 45, 2 (2014), 293–312.

Pandey, S. K., and Paulsen, V. I.
A spectral characterization of *AN* operators.
J. Aust. Math. Soc. 102, 3 (2017), 369–391.